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APPLICATION OF FOURIER-SERIES METHODS AND INTEGRAL EQUATIONS FOR SOLVING NONSTATIONARY NONAXISYMMETRIC HEAT CONDUCTION PROBLEMS FOR BODIES OF REVOLUTION

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We consider the problem of determining nonstationary nonaxisymmetric temperature fields in bodies of revolution appearing on heating by internal heat sources through and due to convective heat exchange with an external medium. The solution of the problem is represented in the form of a Fourier series in an angular coordinate with coefficients being determined by a method of boundary elements. We consider the general case and particular cases of the nonstationary nonaxisymmetric heat conduction problem and determine the asymptotic temperature distributions with a linear variation in time of the heating medium temperature and with heating by moving heat sources.

In a Cartesian coordinate system (x, y, z) the problem of determining the nonstationary temperature field t in a domain D is reduced to solution of the boundary-value problem

$$\Delta t - \frac{1}{a}\frac{\partial t}{\partial \tau} = f, \quad (x, y, z) \in D, \quad \tau > 0, \tag{1}$$

$$Mt = g, \quad (x, y, z) \in \overline{S}, \quad \tau > 0, \quad t\,|_{\tau=0} = t_0, \quad (x, y, z) \in D,$$

where f, t_0, g are given functions; $\overline{S}$ is the boundary of domain D; $Mt = \lambda \partial t/\partial n - \alpha t$; $\partial t/\partial n$ is the normal derivative with respect to the surface S; $g = \alpha T_s$, and T_s is the temperature of the external medium.

1. General Case of the Nonstationary Heat Conduction Problem. We will solve the nonstationary heat conduction problem by the method of the Laplace integral transform with subsequent numerical reversion based on a Fourier series method (modified) [1].

Let us assume that the Laplace transform $F(p)$ of the function $f(t)$, $0 < t < \infty$, is known, with $f(t) \to f_\infty$, when $t \to \infty$, and with the functions $f(0)$ and $f'(0)$ specified (or to be determined). Then, based on [1], the function $f(t)$ can be represented in the form of a rapidly convergent series:

$$f(t) = \frac{1}{l}\exp\left(c\frac{t}{l}\right)\sum_{n=-\infty}^{\infty} F_*(p_n)\exp\left(2\pi n i\frac{t}{l}\right) +$$

$$+ \frac{1}{1-\exp(-c)}\left[f(0) + lf'(0)\left(\frac{t}{l} + \frac{1}{\exp(c)-1}\right) - \exp(-c) f_\infty\right],$$

where

$$F_*(p_n) = F(p_n) - \left(\frac{f'(0)}{p_n} + \frac{f(0)}{p_n^2}\right); \quad p_n = \frac{(c + 2\pi n i)}{l};$$

c is a constant by whose selection it is possible to optimize the convergence of the solution ($\mathrm{Re}(c) > 0$); l is a constant such that $f(t) \approx f(l)$, when $t > l$. We note that direct application of the numerical reversion based on the

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Fourier-series method leads to the necessity of calculating slowly convergent series (with coefficients of the order of $O(n^{-1})$; when considering test examples in [2] this required $\approx$ 1000 terms).

Next we shall examine the case when $f(x, y, z, \tau) = f_\infty(x, y, z)$ and $g(x, y, z, \tau) = g_\infty(x, y, z)$ with $\tau > \tau_0$, and introduce the variable $\theta = a\tau$. Then, on applying the Laplace transform with respect to the variable θ to boundary-value problem (1), we obtain the boundary-value problem concerning the transform $\tilde{t}$ of the function t:

$$(\Delta - p)\,\tilde{t} = \tilde{W} - t_0, \quad (x, y, z) \in D; \quad M\tilde{t} = \tilde{g}, \quad (x, y, z) \in S. \tag{2}$$

Here $\tilde{W}$ and $\tilde{g}$ are the transforms of the functions W and g.

Thus, the problem under consideration is reduced to determination of the functions F from Helmholtz equations of the form

$$(\Delta - k^2)\, F = W, \quad (r, \varphi, z) \in D, \tag{3}$$

which satisfy the boundary conditions

$$\lambda \frac{\partial F}{\partial n} + \alpha F = \alpha\varphi, \quad (r, \varphi, z) \in S.$$

Here and below, the body is related to a cylindrical coordinate system (r, φ, z). The given equation will be solved by employing approaches developed within the framework of the method of boundary elements [3]. For this purpose, we use the integral representation of Eq. (3):

$$F = \int_S (F'\,|_s\, G - F\,|_s\, G')\, ds + \int_D WG dv, \tag{4}$$

where $G = 1/4\pi \exp(-kR)$; $R^2 = r^2 + r_0^2 + (z - z_0)^2 - 2rr_0 \cos(\varphi - \varphi_0)$; G' is the normal derivative along the external normal of the function G at the point (r_0, φ_0, z_0) of the surface S. Substituting the boundary condition into (4) and letting $(r, \varphi, z) \to S$, we derive the following integral equation for determining the functions $F|_s$:

$$F|_s = -\int_S F|_s \left(G' + \frac{\alpha}{\lambda} G\right) ds + \int_D WG dv + \int_S \frac{\alpha}{\lambda} \psi G ds.$$

For the body of revolution we represent the functions W, T_s, $F|_s$, $F'|_s$ in the form of a Fourier series in the angular coordinate:

$$F'|_s = \sum_{n=-\infty}^{\infty} (F'|_\Gamma)_n \exp(in\varphi); \quad F|_s = \sum_{n=-\infty}^{\infty} (F'|_\Gamma)_n \exp(in\varphi);$$

$$W = \sum_{n=-\infty}^{\infty} W_n \exp(in\varphi); \quad T_s = \sum_{n=-\infty}^{\infty} T_{s_n} \exp(in\varphi),$$

where Γ is the generatrix of the surface S. Substituting these formulas into (4), we obtain the integral representation

$$F_n = \int_\Gamma r_0 (F_n'|_\Gamma G_n - F_n|_\Gamma G_n')\, dl + \int_D r_0 \psi f_n dD,$$

where F_n is the n-th coefficient in the Fourier series for the function F:

$$F = \sum_{n=-\infty}^{\infty} F_n \exp(in\varphi), \quad G_n = \frac{1}{4\pi} \int_0^{2\pi} \frac{\exp(-k\rho)}{\rho} \cos(n\eta)\, d\eta.$$

Here $\rho^2 = r^2 + r_0^2 + (z - z_0)^2 - 2rr_0 \cos(\eta)$. In this case the integral boundary-value equation takes the form

$$F_n|_\Gamma = -\int_\Gamma r_0 F_n|_\Gamma \left(G_n' + \frac{\alpha}{\lambda} G_n\right) dl + \int_\Gamma r_0 \psi G \frac{\alpha}{\lambda}\, dl. \tag{5}$$

The value of T^∞ can be calculated with the help of Eq. (5), when $k = 0$.

2. Determination of the Asymptotic Temperature Distribution. Let us consider the case, frequently encountered in practice, in which the temperature of the external medium is a linearly time-varying function $T_s = \alpha_0 + \theta\alpha_1$. As is known, in this case at sufficiently large time instants the temperature can be represented approximately as

$$t = t_0 + \theta t_1 \,, \quad \Delta t_1 = 0 \,, \quad \Delta t_0 = t_1 \,.$$

In the case of a three-dimensional problem, the integral representations for the given functions are

$$T_1 = C_1 \int_S \left(T_1' |_s \frac{1}{R} - T_1 |_s \frac{\partial}{\partial n_0} \left[\frac{1}{R} \right] \right) ds \,,$$

$$T_0 = C_1 \int_S \left(T_0' |_s \frac{1}{R} - T_0 |_s \frac{\partial}{\partial n_0} \left[\frac{1}{R} \right] \right) ds + \frac{C_1}{2} \int_S \left(T_1' |_s R - T_1 |_s \frac{\partial R}{\partial n_0} \right) ds \,, \tag{6}$$

where $T_j' = \partial T_j / \partial n_0$, $j = 0.1$; $\partial/\partial n$ and $\partial/\partial n_0$ are the normal derivatives to the surface S with respect to the coordinates (r, φ, z) and (r_0, φ_0, z_0), respectively; $C_1 = 1/(4\pi)$, if the point of (r, φ, z) lies inside of the region bounded by the surface S, and $C_1 = -1/(4\pi)$ for the external region.

For the case of the body of revolution under consideration, we represent the prescribed and desired functions in the form of a Fourier-series expansion in the angular coordinate:

$$\alpha_0 = \sum_{n=-\infty}^{\infty} a_n \exp(in\varphi) \,, \quad \alpha_1 = \sum_{n=-\infty}^{\infty} b_n \exp(in\varphi) \,,$$

$$T_0 = \sum_{n=-\infty}^{\infty} A_n \exp(in\varphi) \,, \quad T_1 = \sum_{n=-\infty}^{\infty} B_n \exp(in\varphi) \,,$$

where $a_n = a_n(r, z)$, b_n are the prescribed functions; A_n and B_n are unknowns. Substituting these formulas into (6) and performing transformations, we obtain the integral representations

$$A_n = C_1 \int_\Gamma r_0 \left(A_n' |_\Gamma f_n - A_n |_\Gamma \frac{\partial f_n}{\partial n_0} \right) d\Gamma \,,$$

$$B_n = C_1 \int_\Gamma r_0 \left(B_n' |_\Gamma f_n - B_n |_\Gamma \frac{\partial f_n}{\partial n_0} \right) d\Gamma + \frac{C_1}{2} \int_\Gamma r_0 \left(A_n' |_\Gamma g_n - A_n |_\Gamma \frac{\partial g_n}{\partial n_0} \right) d\Gamma \,,$$

where

$$f_n = \int_0^{2\pi} \frac{\cos(n\varphi)}{R_0} d\varphi \,, \quad g_n = \int_0^{2\pi} R_0 \cos(n\varphi)\, d\varphi = -\frac{m\tilde{\rho}^2}{4} (f_{n+1} + \mu f_n + f_{n-1}) \,.$$

We determined the functions f_n using the recurrence relations

$$f_{n+2} = 2\mu \frac{n+1}{n+1{,}5} f_{n+1} - \frac{n+0{,}5}{n+1{,}5} f_n \,, \quad n > 0 \,,$$

$$f_n = 4\,(D_n E(m) + Q_n K(m))/\tilde{\rho} \,, \quad D_0 = 0 \,, \quad D_1 = 1 - \mu \,, \quad Q_0 = 1 \,, \quad Q_1 = \mu \,, \quad n = 0 \,, \ 1 \,,$$

where $\mu = 2/m - 1$, $m = 4rr_0/\tilde{\rho}^2$, $\tilde{\rho}^2 = \bar{z}^2 + (r - r_0)^2$, $E(m)$ and $K(m)$ are total elliptic integrals of the second- and first-order, respectively. We note that

$$\frac{\partial g_n}{\partial r} = r f_n - \frac{r_0}{2} (f_{n+1} + f_{n-1}) \,, \quad \frac{\partial g_n}{\partial z} = \bar{z} f_n \,, \quad \bar{z} = z - z_0 \,.$$

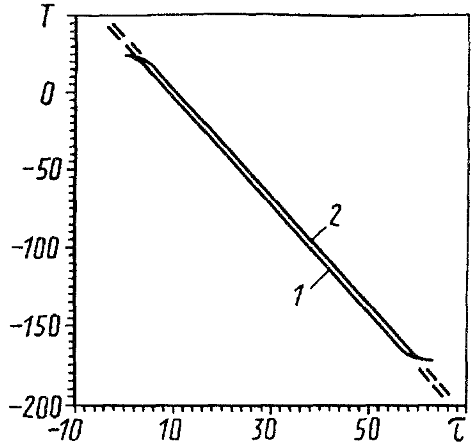

Fig. 1. Time variation of the temperature T, °C, in a cylindrical shell supported by a frame: 1) temperature at the point with coordinates (2, 0, 15) and 2) temperature at the point with coordinates (2, 18, 0.01). Solid lines refer to the solution constructed by the algorithm under item 1; dashed lines refer to the asymptotic solution (see item 2).

Based on Eq. (1), the boundary conditions for the functions introduced above will be written in the form (the case is considered in which the body is heated by the external medium T_s according to Newton's law)

$$\lambda \frac{\partial A_n}{\partial n} + \alpha (A_n - b_n) = 0 , \quad \lambda \frac{\partial B_n}{\partial n} + \alpha (B_n - a_n) = 0 . \tag{7}$$

Using integral representations (6) and boundary conditions (7), we obtain integral equations for the functions A_n and B_n:

$$A_n = - C_1 \int_\Gamma r_0 A_n \left[\frac{\alpha}{\lambda} f_n + \frac{\partial f_n}{\partial n_0} \right] ds + C_1 \int_\Gamma r_0 b_n \frac{\alpha}{\lambda} f_n ds ,$$

$$B_n = - C_1 \int_\Gamma r_0 B_n \left[\frac{\alpha}{\lambda} f_n + \frac{\partial f_n}{\partial n_0} \right] ds + C_1 \int_\Gamma r_0 a_n \frac{\alpha}{\lambda} f_n ds -$$

$$- \frac{C_1}{2} \int_\Gamma r_0 A_n \left[\frac{\alpha}{\lambda} g_n + \frac{\partial g_n}{\partial n_0} \right] ds + \frac{C_1}{2} \int_\Gamma r_0 b_n \frac{\alpha}{\lambda} g_n ds .$$

3. Heating of a Body by Circularly Moving Sources. Now we will consider a body of revolution heated by heat sources moving at a constant velocity in a circular direction. Let us take the case when $T_s = T_s(r, z, \varphi - \omega\tau)$, i.e., in a moving coordinate system $\varphi_1 = \varphi - \omega\tau$ the tempearture of the external medium is independent of time. Here ω is the angular velocity of the source.

Let us introduce the function $T = T(r, z, \varphi_1, \tau)$. For this function to be defined we obtain from relations (1) the boundary-value problem

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi_1^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega}{a} \frac{\partial}{\partial \varphi_1} - \frac{1}{a} \frac{\partial}{\partial \tau} \right) T = 0 ,$$

$$\lambda \frac{\partial T}{\partial n} - \alpha T = - \alpha T_s , \quad T|_{\tau=0} = 0 , \tag{8}$$

whose solution will be sought for $\tau \to \infty$ (steady-state regime) in the form of a Fourier series:

$$T = \sum_{k=-\infty}^{\infty} T_k (r , z) \exp (ik\varphi) = T_0 (r , z) + 2 \sum_{k=1}^{\infty} \mathrm{Re} \, (T_k \exp (ik\varphi)) .$$

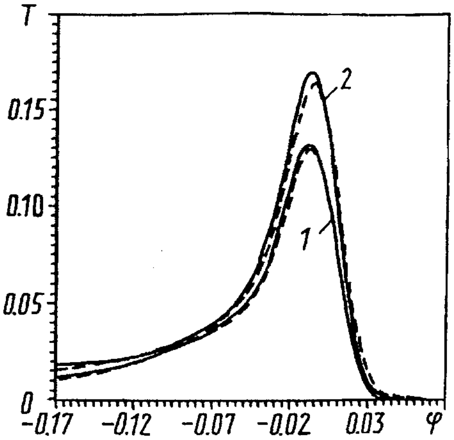

Fig. 2. Variation of the dimensionless temperature in a cylindrical shell with the angle φ: 1) $\gamma = 2500\ \mathrm{m}^{-2}$, 2) $1000\ \mathrm{m}^{-2}$ (dashed lines refer to the temperature in an infinite plate at the same parameters).

Substituting this series into relations (8), we see that the functions T_k ($k = 0, \pm 1, \ldots$) are determined from the equations

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k^2}{r^2} + \frac{\partial^2}{\partial z^2} - \gamma_k^2\right) T_k = 0 \tag{9}$$

and that they satisfy the boundary conditions

$$\lambda \frac{\partial T_k}{\partial n} - \alpha T_k = -\alpha T_{sk}.$$

Here $\gamma_k^2 = -i(\omega/a)k$, $\mathrm{Re}(\gamma_k) > 0$, T_{sk} are the coefficients of the Fourier-series expansion of the function T_s, i.e.,

$$T_{sk} = \frac{1}{2\pi}\int_0^{2\pi} T_s \exp(-ik\varphi)\, d\varphi .$$

The solution of Eq. (9) will be determined by the method of boundary elements using the representation

$$T_k = \int_S r_0 \left(T_k'|_s G_k - T_k|_s G_k'\right) ds, \quad G_k' = \frac{\partial G_k}{\partial n_0}, \quad T_k' = \frac{\partial T_k}{\partial n}, \tag{10}$$

G_k is the fundamental solution of Eq. (9), i.e.,

$$G_k = -\frac{r_0}{2\pi}\int_0^{2\pi} \cos(k\eta)\frac{\exp(-\gamma_k\rho)}{\rho}\, d\eta .$$

4. Results of Numerical Calculations. Computer programs have been developed on the basis of the foregoing solutions. We will give the results of calculations of temperatures for some particular cases that will allow the effectiveness of the approach to be judged.

A. We calculated numerically the temperature of a structure element having the form of a cylindrical shell $R_1 < r < R_2$, $|z| < L$ supported at $r = R_2$, $|z| < 0.01$ m by a frame of height 0.13 m. The heat transfer coefficients were taken to be $\alpha = 65\ \mathrm{kcal/(m^2 \cdot deg)}$ at $r = R_1$ and $\alpha = 5\ \mathrm{kcal/(m^2 \cdot deg)}$, when $r > R_1$. In the calculations it was assumed that $R_1 = 2$ m, $R_2 = 2.05$ m, $L = 0.15$ m, $\lambda = 9.8\ \mathrm{kcal/(m \cdot deg)}$, $a = 0.0000045\ \mathrm{m/sec^2}$. The cross sections $z = \pm L$ were taken to be thermally insulated. The temperature of the external medium was chosen in the form $T_s = f\Phi$, $f = \tilde{v}$, $\tilde{v} = -0.00364$ deg/sec, where

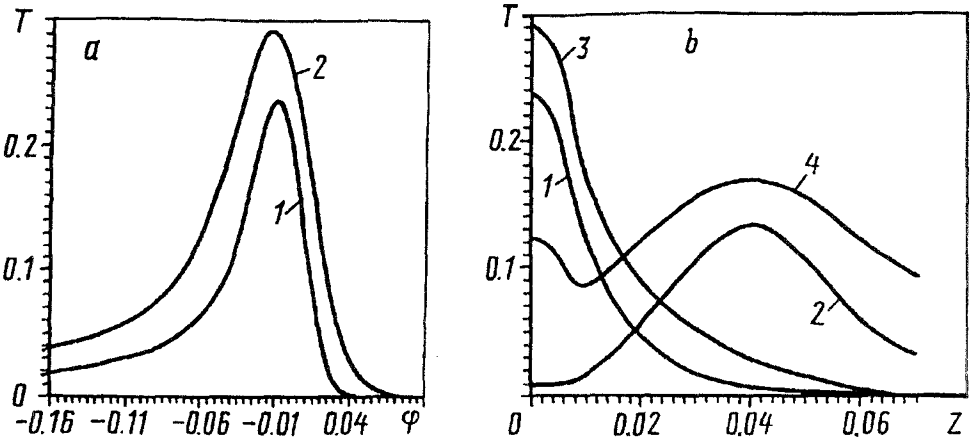

Fig. 3. Variation of the dimensionless temperature in a circularly welded cylindrical shell with the angle φ (a) and z (b); a: 1) $\gamma = 2500\ \mathrm{m}^{-2}$, 2) 1000; b: 1, 3) heating by one source with the center at the point (R_2, 0) when $\gamma = 2500\ \mathrm{m}^{-2}$ (1) and 1000 (3); 2, 4) heating by two sources with the centers at the point (R_2, ±0.04) when $\gamma = 2500\ \mathrm{m}^{-2}$ (2) and $1000\ \mathrm{m}^{-2}$ (4).

$$\Phi = \begin{cases} \dfrac{\tau^2 - 0{,}5\tau\tau_0^2}{0{,}5\tau_0^2} - \gamma_0 \dfrac{\tau^2 - \tau\tau_0}{0{,}25\tau\tau_0}, & \tau \le \tau_0, \\ 1, & \tau > \tau_0. \end{cases}$$

We note that for these boundary conditions there is a problem of bilateral freezing of the structure element on the assumption that the temperature of the external medium varies from 0 to the prescribed one $T_s^0 = -173°\mathrm{C}$, at time τ_0 according to a quadratic law.

Figure 1 shows the change in the temperature at the points with coordinates (2, 0, 15) (curves 1) and (2, 18, 0.01) (curves 2) in time at $\gamma_0 = 0.5$, $\tau_0 = 54{,}000$ sec. The solid lines represent the results obtained by the general procedure, while the dashed lines illustrate the results obtained with allowance for the linear time variation in the temperature of the external medium (asymptotic solution).

B. Using the given algorithm, we calculated the steady-state temperature in a cylinder $R_1 < r < R_2$, $-L < z < L$, heated by means of heat exchange by a circularly moving external medium whose temperature is described by the Gauss normally circular law:

$$T_s = T_0 \exp\left(-\gamma\left(z_2^2 + R^2\varphi^2\right)\right),$$

where $T_0 = \mathrm{const}$ at $r = R_2$ and $T_0 = 0$, when $r < R_2$; γ is the coefficient of the concentration of heating. For the case of local heating, when $\gamma\pi^2 R_2^2 << 1$, the coefficients of the Fourier-series expansion of the external medium temperature are written as

$$T_{sk} = T_0 \frac{\exp(-\gamma z^2)}{2\pi\gamma R_2} \exp\left(-\frac{k^2}{4\gamma R_2^2}\right), \quad k = 0, \pm 1, \ldots$$

In the calculations it was assumed that $a = 1\ \mathrm{m/sec}^2$, $\lambda = 1\ \mathrm{kcal/(m \cdot deg)}$, $R_1 = 0.98$ m, $\alpha = 20\ \mathrm{kcal/(m^2 \cdot deg)}$, $R_2 = 1$ m, $v = 0.02$ m/sec, $L = 0.07$ m.

Figure 2 illustrates the change in temperature T as a function of φ. Curves 1 refer to $\gamma = 2500\ \mathrm{m}^{-2}$, curves 2 to $\gamma = 1000\ \mathrm{m}^{-2}$. The dashed lines denote the change in temperature for $R_{1,2} \to \infty$, $R_2 - R_1 = 0.02$ m in an infinite plate for the same parameters obtained from the exact formulas of [4]. We can see from the figure that in the given case the curvature has virtually no effect on the temperature distribution.

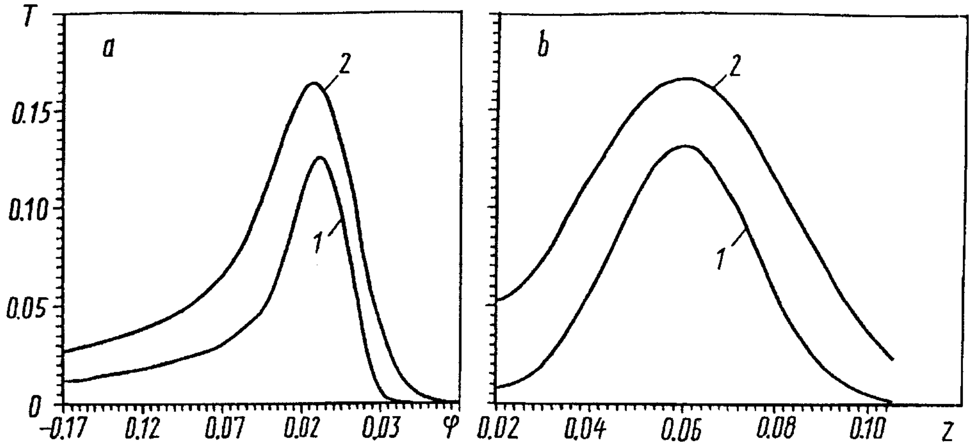

Fig. 4. Variation of the dimensionless tempearture in a cylindrical shell supported by a frame with the angle φ (a) and z (b): 1) $\gamma = 2500\ \mathrm{m}^{-2}$, 2) 1000.

We calculated the temperature in a circularly welded cylindrical shell heated by one and two moving heat sources (investigations were carried to select heat treatment regimes to remove residual stresses). The boundary of the shell was described by the relations

$$r = R_1\,,\ \ |z| < L\,;\ \ z = \pm L\,,\ \ R_1 < r < R_2\,;\ \ r = R_2\,,\ \ a_1 < |z| < L\,;$$

$$|z| < a_1\,,\ r = R_2 + d/2\,(1 + \cos(\pi z/a_1))\,.$$

The results of the calculations of the temperature (at the parameters given above for the case of the cylindrical shell and $L = 0.07$ m, $a_1 = 0.01$ m, $d = 0.005$ m) on the outer side of the shell in the cross sections $z = 0$ and $\varphi = 0$ are presented in Fig. 3. As is seen from the figure, in the case of heating by two moving sources the maximum temperature turned out to be substantially smaller than that for one source, due to the difference in the shapes of the surfaces of the heated sources.

Figure 4 gives the calculated results for the temperature in a cylindrical shell supported by a frame on the external side (at the above-given parameters) and heated by two moving heat sources with centers at the point $(R_2, \pm C)$, $C = 0.06$ m.

As is seen from the figures, in the present case the difference in the geometry of the shells has a little effect on the temperature in the heated regions.

NOTATION

T, temperature; τ, time; Δ, Laplace operator; λ, thermal conductivity coefficient; α, heat transfer coefficient; ω, angular velocity of source; γ, coefficient of heating concentration; (x, y, z), Cartesian coordinate system; (r, φ, z), cylindrical coordinate system; a, thermal diffusivity coefficient; v, linear velocity of source.

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